LOCAL DEFORMATION RINGS FOR 2-ADIC REPRESENTATIONS OF $G_{\mathbb{Q}_l}$, $l \neq 2$.

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Let l and p be distinct primes. Let $L/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$, uniformiser ϖ and residue field k. Let $F/\mathbb{Q}_l$ be a finite extension with absolute Galois group G_F, inertia group I_F, and wild inertia group P_F. Let $\tilde{P}_F$ be the kernel of the maximal pro-l quotient of I_F. Let q be the order of the residue field of F. We assume that L contains all $(q^2-1)th$ roots of unity. Choose a pro-generator σ of $I_F/\tilde{P}_F$ and $\phi \in G_F/\tilde{P}_F$ lifting the arithmetic Frobenius element of G_F/I_F. Then we have the relation

$$\phi\sigma\phi^{-1} = \sigma^q. \tag{1}$$

If $\overline{\rho} : G_F \to GL_2(k)$ is a continuous homomorphism, let $R^{\square}_{\overline{\rho}}$ be the universal framed deformation ring for $\overline{\rho}$ parametrising lifts with coefficients in $\mathcal{O}$-algebras. By [Sho16a] Theorem 2.5, $R^{\square}_{\overline{\rho}}$ is a reduced, $\mathcal{O}$-flat complete intersection ring of relative dimension 4 over $\mathcal{O}$.

If $\tau : I_F \to GL_2(L)$ is a continuous semisimple representation that extends to G_F, let $R^{\square}_{\overline{\rho}}(\tau)$ be the maximal reduced, p-torsion free quotient of $R^{\square}_{\overline{\rho}}$ such that, for every $\mathcal{O}$-algebra homomorphism $x : R^{\square}_{\overline{\rho}} \to \overline{L}$, the corresponding representation $\rho_x : G_F \to GL_2(\overline{L})$ satisfies $(\rho_x|_{I_F})^{ss} \cong \tau$.

The goal of this appendix is to prove:

Theorem 0.1. *For any $\overline{\rho}$ and τ as above, the ring $R^{\square}_{\overline{\rho}}(\tau)$ is either Cohen–Macaulay or zero.*

If $p > 2$, then this is the content of section 5.5 of [Sho16b]. If $p = 2$ and $\overline{\rho}|_{\tilde{P}_F}$ is non-scalar, then the proof of proposition 5.1 of [Sho16b] shows that $R^{\square}_{\overline{\rho}}$ is a completed tensor product of deformation rings of characters, all of whose irreducible components are formally smooth, and that $R^{\square}_{\overline{\rho}}(\tau)$ is an irreducible component of $R^{\square}_{\overline{\rho}}$; thus $R^{\square}_{\overline{\rho}}(\tau)$ is formally smooth in this case. From now on, then, we assume that $p = 2$ and that $\overline{\rho}|_{\tilde{P}_F}$ is scalar; by twisting, we may and do assume that $\overline{\rho}|_{\tilde{P}_F}$ is trivial. In this case, we may list the semisimple inertial types τ for which $R^{\square}_{\overline{\rho}}(\tau)$ may be non-zero. They are determined by the eigenvalues of $\tau(\sigma)$, which must be of 2-power order and either fixed or interchanged by raising to the power q. Writing $a = v_2(q-1)$ and $b = v_2(q^2-1)$, if $R^{\square}_{\overline{\rho}}(\tau)$ is non-zero then either

- $\tau = \tau_\zeta$ is the inertial type in which the eigenvalues of $\tau(\sigma)$ are both equal to an 2^ath root of unity, ζ;
- $\tau = \tau_{\zeta_1,\zeta_2}$ is the inertial type in which the eigenvalues of $\tau(\sigma)$ are equal to distinct 2^ath roots of unity ζ_1 and ζ_2;
- $\tau = \tau_\xi$ is the inertial type in which the eigenvalues of $\tau(\sigma)$ are equal to ξ and ξ^q for ξ an 2^bth root of unity with $\xi \neq \xi^q$ (equivalently, with ξ not an 2^ath root of unity).

We also give a version with fixed determinant:

Corollary 0.2. *If ψ is any lift of $\det \overline{\rho}$ to $\mathcal{O}^\times$ such that $\psi|_{I_F} = \det \tau$, let $R^{\square,\psi}_{\overline{\rho}}(\tau)$ be the universal framed deformation ring with determinant ψ and type τ. Then $R^{\square,\psi}_{\overline{\rho}}(\tau)$ is Cohen–Macaulay.*

Proof. By Theorem 0.1, $R^{\square}_{\overline{\rho}}(\tau)$ is Cohen–Macaulay. If we impose a single equation additional $\det \rho(\phi) = \psi(\phi)$, then the ring will still be Cohen–Macaulay provided that $\det \rho(\phi) - \psi(\phi)$ is a non-zerodivisor — in other words, that it doesn't vanish on any irreducible components of $\operatorname{Spec} R^{\square}_{\overline{\rho}}(\tau)$. This is the case, since the action of $\mathbb{G}_m^\wedge$ on $\operatorname{Spec} R^{\square}_{\overline{\rho}}(\tau)$ given by making unramified twists preserves irreducible components but varies the determinant. $\square$

Let $\mathcal{X}$ be the affine $\mathcal{O}$-scheme whose R points, for an $\mathcal{O}$-algebra R, are pairs

$$\{(\Sigma, \Phi) \in GL_2(R) \times GL_2(R) : \Phi\Sigma = \Sigma^q \Phi\}.$$

Then $\mathcal{X}$ is a reduced, $\mathcal{O}$-flat complete intersection of relative dimension 4 over $\operatorname{Spec} \mathcal{O}$ by the proof of Theorem 2.5 of [Sho16a]. Let $\mathcal{A}$ be the coordinate ring of $\mathcal{X}$. We write $\Sigma = \begin{pmatrix} 1+A & B \\ C & 1+D \end{pmatrix}$ and $\Phi = \begin{pmatrix} P & Q \\ R & T-P \end{pmatrix}$, so that $\mathcal{A}$ is a quotient of

$$\mathcal{S} = \mathcal{O}[A, B, C, D, P, Q, R, T][(\det \Sigma)^{-1}, (\det \Phi)^{-1}].$$

For any continuous $\overline{\rho} : G_F \to GL_2(k)$, the pair of matrices $\overline{\rho}(\sigma)$ and $\overline{\rho}(\phi)$ give rise to a closed point of $\mathcal{X}$, and so a maximal ideal $\mathfrak{m}$ of $\mathcal{A}$. Then $R^{\square}_{\overline{\rho}} = \mathcal{A}^\wedge_{\mathfrak{m}}$. If $\mathcal{C}$ is a conjugacy class in $GL_2(\overline{L})$, then there is a unique irreducible component of $\mathcal{X}$ such that, for a dense set of geometric points of that component, the corresponding matrix Σ has conjugacy class $\mathcal{C}$. This provides a bijection between the irreducible components of $\mathcal{X}$ and the conjugacy classes of $GL_2(\overline{L})$ that are preserved under the q-power map (by [Sho16a] Proposition 2.6). If τ is one of the above inertial types then we write $\mathcal{X}(\tau)$ for the union of those irreducible components corresponding to conjugacy classes with the same characteristic polynomial as $\tau(\sigma)$, with the reduced subscheme structure, and $\mathcal{A}(\tau)$ for its coordinate ring. Note that, since $\mathcal{X}$ is $\mathcal{O}$-flat and $\mathcal{X}(\tau)$ is an irreducible component of $\mathcal{X}$, $\mathcal{X}(\tau)$ is also $\mathcal{O}$-flat, so that $\mathcal{A}(\tau)$ is ϖ-torsion free.

Lemma 0.3. *If $\tau = \tau_\zeta, \tau_{\zeta_1,\zeta_2}$, or τ_ξ, then $\mathcal{A}(\tau)^\wedge_{\mathfrak{m}} = R^{\square}_{\overline{\rho}}(\tau)$.*

Proof. Since $\mathcal{A}$ is $\mathcal{O}$-flat and $\mathcal{A}(\tau)$ is the quotient of $\mathcal{A}$ by an intersection of minimal prime ideals, it is also $\mathcal{O}$-flat. Thus $\mathcal{A}(\tau)^\wedge_{\mathfrak{m}}$ is also $\mathcal{O}$-flat, by flatness of localisation and completion. Since $\mathcal{A}(\tau)$ is of finite type over a DVR it is Nagata by [Sta17, Tag 0335]. Since $\mathcal{A}(\tau)$ is reduced, the completion $\mathcal{A}(\tau)^\wedge_{\mathfrak{m}}$ is also reduced by [Sta17, Tag 07NZ]. The composite map $\mathcal{A} \to R^{\square}_{\overline{\rho}} \twoheadrightarrow R^{\square}_{\overline{\rho}}(\tau)$ factors through a map $\mathcal{A}(\tau) \to R^{\square}_{\overline{\rho}}(\tau)$, since any function in $\mathcal{A}$ that vanishes on all $\overline{L}$-points of type τ must vanish in $R^{\square}_{\overline{\rho}}(\tau)$ by definition. Thus we get a surjection $\mathcal{A}(\tau)^\wedge_{\mathfrak{m}} = \mathcal{A}(\tau) \otimes_{\mathcal{A}} R^{\square}_{\overline{\rho}} \twoheadrightarrow R^{\square}_{\overline{\rho}}(\tau)$. However, since $\mathcal{A}(\tau)^\wedge_{\mathfrak{m}}$ is reduced and $\mathcal{O}$-torsion free, and has the property that every $\overline{L}$-point gives a Galois representation of type τ, this map is an isomorphism by the definition of $R^{\square}_{\overline{\rho}}(\tau)$. $\square$

Let $\overline{\mathcal{S}} = \mathcal{S} \otimes_{\mathcal{O}} k$, $\overline{\mathcal{A}} = \mathcal{A} \otimes_{\mathcal{O}} k$, and $\overline{\mathcal{X}} = \operatorname{Spec} \overline{\mathcal{A}}$. Then the irreducible components of $\overline{\mathcal{X}}$ are in bijection with the conjugacy classes of $GL_2(\overline{k})$ that are stable under the

q-power map (again by [Sho16a] Proposition 2.6). Let $\overline{\mathcal{X}}_1$ be the irreducible component corresponding to the trivial conjugacy class — this is just the locus where $\Sigma = 1$ — and let $\overline{\mathcal{X}}_N$ be that corresponding to the non-trivial unipotent conjugacy class (we give the irreducible components the reduced subscheme structure). Let I_1 and I_N be the prime ideals of $\overline{\mathcal{S}}$ cutting out $\overline{\mathcal{X}}_1$ and $\overline{\mathcal{X}}_N$; these correspond to minimal primes of $\overline{\mathcal{A}}$. If τ is one of the above inertial types, then we write $I(\tau)$ for the ideal of $\overline{\mathcal{S}}$ cutting out $\mathcal{A}(\tau) \otimes_{\mathcal{O}} k$.

Lemma 0.4. *The ideals I_1 and I_N have generators*

$$\begin{aligned} I_1 &= (A, B, C, D) \\ I_N &= (A^2 + BC, CQ + BR, T, A + D). \end{aligned}$$

Proof. The presentation for I_1 is obvious. For I_N, the condition that Σ is unipotent gives $A + D \in I_N$ and $A^2 + BC \in I_N$. If $N = \Sigma - 1$, then the relation $\Phi\Sigma = \Sigma^q\Phi$ becomes $\Phi N = qN\Phi = N\Phi$ (since we are working mod 2), which implies that $CQ + BR = 0$. At any closed point of $\overline{\mathcal{X}}_N$ where $N \neq 0$, the eigenvalues of Φ must be in the ratio $1 : q = 1 : 1$, and so $T = 0$. As such closed points are dense on $\overline{\mathcal{X}}_N$, we see that $T \in I_N$. Therefore

$$(A^2 + BC, CQ + BR, T, A + D) \subset I_N.$$

The ideal $I = (A^2 + BC, CQ + BR, T, A + D)$ is prime of dimension 4; indeed, $\mathcal{S}/I$ is isomorphic to a localisation of

$$\frac{k[A, B, C]}{(A^2 + BC)}[P, Q, R]/(CQ + BR)$$

which is easily seen to be a 4-dimensional domain. Thus $I \subset I_N$ are prime ideals of $\overline{\mathcal{S}}$ of the same dimension, and so must be equal. □

Proposition 0.5. *Let $\tau = \tau_\xi$. Then $I(\tau) = I_N$.*

Proof. Write $\eta = \xi + \xi^q - 2$. The condition that Σ has characteristic polynomial $(X - \xi)(X - \xi^q)$ shows that, on $\mathcal{X}(\tau)$, we have the equations

$$\begin{aligned} A + D &= \eta \\ A(A - \eta) + BC &= \eta. \end{aligned}$$

Using the first of these, we replace D by $\eta - A$ everywhere. Now, if x is an $\overline{L}$-point of $\mathcal{X}(\tau)$ corresponding to a pair of matrices (Σ_x, Φ_x), then Φ_x exchanges the ξ and ξ^q eigenspaces of Σ_x and so must have trace zero. Therefore on $\mathcal{X}(\tau)$ we have the equation

$$T = 0.$$

Lastly, by the Cayley–Hamilton theorem, and the fact that

$$X^q \equiv \xi + \xi^q - X \mod (X - \xi)(X - \xi^q),$$

we see that $\Sigma^q = \begin{pmatrix} 1 + \eta - A & -B \\ -C & 1 + A \end{pmatrix}$ on $\mathcal{X}(\tau)$. Equating matrix entries in the relation $\Phi\Sigma = \Sigma^q\Phi$, and noting that $T = 0$, we obtain one new equation

$$(2A - \eta)P + BR + CQ = 0.$$

Thus, letting

$$J = (A + D, T, A(A - \eta) + BC - \eta, (2A - \eta)P + BR + CQ)$$

we obtain a surjection $\mathcal{S}/J \twoheadrightarrow \mathcal{A}(\tau)$, and therefore a surjection

$$\overline{\mathcal{S}}/J \twoheadrightarrow \overline{\mathcal{A}}(\tau).$$

As η is divisible by ϖ, we see that $J + (\varpi) = I_N$, and so we have a surjection $\overline{\mathcal{S}}/I_N \twoheadrightarrow \overline{\mathcal{A}}(\tau)$. This must be an isomorphism since $\overline{\mathcal{S}}/I_N$ is a 4-dimensional domain and $\overline{\mathcal{A}}(\tau)$ is a non-zero 4-dimensional ring. Therefore $I_N = I(\tau)$ as required. □

For the remaining types the following lemmas will be useful. If R is a noetherian ring, $\mathfrak{p}$ is a minimal prime of R, and M is a finitely-generated R-module, let $e_R(M, \mathfrak{p}) = l_{R_\mathfrak{p}}(M_\mathfrak{p})$ (this is a special case of the Hilbert–Samuel multiplicity).

Lemma 0.6. *Let $f : R \to S$ be a surjection of equidimensional rings of the same dimension, and suppose that R is S1 and Nagata. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of R. Suppose that, for $i = 1, \ldots, n$, there is a maximal ideal $\mathfrak{m}_i$ of S such that $\mathfrak{p}_i \subset \mathfrak{m}_i$ but $\mathfrak{p}_j \not\subset \mathfrak{m}_i$ for $i \neq j$. If, for each i, we have*

$$e_R(R, \mathfrak{p}_i) \leq e_{S^\wedge_{\mathfrak{m}_i}}(S^\wedge_{\mathfrak{m}_i}, \mathfrak{q}_i)$$

for some minimal prime $\mathfrak{q}_i$ of $S^\wedge_{\mathfrak{m}_i}$, then f is an isomorphism.

Remark 0.7. For those primes $\mathfrak{p}_i$ such that $e_R(R, \mathfrak{p}_i) = 1$ — which is all of them if R is reduced — the required inequality is implied simply by the existence of the $\mathfrak{m}_i$.

Proof. Since R is S1, every associated prime of S is minimal and so, by [Sta17, Tag 0311], it is enough to show that f induces an isomorphism $f_{\mathfrak{p}_i} : R_{\mathfrak{p}_i} \to S_{\mathfrak{p}_i}$ for each i. Since f is surjective and $R_{\mathfrak{p}_i}$ is artinian, it is enough to show that $e_R(R, \mathfrak{p}_i) \leq e_R(S, \mathfrak{p}_i)$. Let $i \in \{1, \ldots, n\}$. Choose $\mathfrak{m}_i$ and $\mathfrak{q}_i$ as in the hypotheses of the lemma. It is enough to show that for each i,

$$e_R(S, \mathfrak{p}_i) = e_{S^\wedge_{\mathfrak{m}_i}}(S^\wedge_{\mathfrak{m}_i}, \mathfrak{q}_i).$$

Since $\mathfrak{m}_i$ contains a unique minimal prime of R, after localising at $\mathfrak{m}_i$ we may assume that $R \to S$ is a local map of local rings, and that $\mathfrak{p}_i$ is the unique minimal prime of R, and drop i from the notation. The hypothesis that R and S are equidimensional of the same dimension implies that $\mathfrak{p}S$ is the unique minimal prime of S, which we also denote by $\mathfrak{p}$. We have $e_R(S, \mathfrak{p}) = e_S(S, \mathfrak{p})$ since both are just the length of $S_\mathfrak{p}$. Since $S \to S^\wedge$ is flat and $S^\wedge/\mathfrak{p} = (S/\mathfrak{p})^\wedge$ is reduced because R (and hence S) is Nagata, [Sta17, Tag 02M1] implies that $e_S(S, \mathfrak{p}) = e_{S^\wedge}(S^\wedge, \mathfrak{q})$. So

$$e_R(S, \mathfrak{p}) = e_S(S, \mathfrak{p}) = e_{S^\wedge}(S^\wedge, \mathfrak{q}) \geq e_R(R, \mathfrak{p})$$

as required. □

The S1 condition holds, in particular, if R is reduced or Cohen–Macaulay, while the Nagata condition holds if R is of finite type over a field or DVR.

Proposition 0.8. *Let $\tau = \tau_\zeta$. Then*

$$\begin{aligned} I(\tau) &= I_N \cap I_1 \\ &= (A + D, AT, BT, CT, A^2 + BC, BR + CQ). \end{aligned}$$

Proof. For simplicity, we twist so that $\zeta = 1$. Write $N = \Sigma - 1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. On $\mathcal{A}(\tau)$, Σ has characteristic polynomial $(X-1)^2$, and so the equations

$$A + D = 0$$
$$A^2 + BC = 0$$

hold on $\mathcal{A}(\tau)$. Moreover, since $(\Sigma - 1)^2 = 0$ on $\mathcal{A}(\tau)$, by the Cayley–Hamilton theorem we have that $\Sigma^q = 1 + q(\Sigma - 1) = 1 + qN$ on $\mathcal{A}(\tau)$. The equation $\Phi\Sigma = \Sigma^q\Phi$ becomes $\Phi N = qN\Phi$, and comparing matrix entries we get equations

$$qBR - CQ + (q-1)AP = 0$$
$$(q+1)QA + B(qT - (q+1)P) = 0$$
$$(q+1)RA + C(T - (q+1)P)) = 0$$
$$qCQ - BR + (q-1)A(P-T) = 0.$$

Summing the first and fourth of these gives $(q-1)(BR + CQ + A(2P - T)) = 0$; since $\mathcal{A}(\tau)$ is $(q-1)$-torsion free, we deduce that

$$BR + CQ + A(2P - T) = 0$$

in $\mathcal{A}(\tau)$ and can replace the fourth of the above equations by this.

The ideal cutting out $\mathcal{A}(\tau)$ therefore contains the ideal

$$J = \big(A + D, A^2 + BC, qBR - CQ + (q-1)AP, (q+1)QA + B(qT - (q+1)P), \\ (q+1)RA + C(T - (q+1)P), CQ + BR + A(2P - T)\big).$$

Now, the image of J in $\overline{\mathcal{S}}$ is

$$(A + D, A^2 + BC, BR + CQ, BT, CT, BR + CQ + AT)$$

which is equal to $(A + D, A^2 + BC, BR + CQ) + I_1 \cap (T) = I_N \cap I_1$. Therefore there is a surjection

$$f : \overline{\mathcal{S}}/(I_N \cap I_1) \twoheadrightarrow \overline{\mathcal{A}}(\tau).$$

Write $\tilde{R} = \overline{\mathcal{S}}/(I_N \cap I_1)$. Then $\tilde{R}$ is reduced with two minimal primes, which we also call I_N and I_1. Let $\rho_1 : G_F \to GL_2(\mathcal{O})$ be diagonal unramified with distinct eigenvalues of Frobenius, and let $\rho_N : G_F \to GL_2(\mathcal{O})$ send $\sigma \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\phi \mapsto \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$. Let $\mathfrak{m}_1$ and $\mathfrak{m}_N$ be the corresponding maximal ideals of $\overline{\mathcal{A}}(\tau)$. Then $I_1 \subset \mathfrak{m}_1$, $I_N \not\subset \mathfrak{m}_1$, $I_1 \not\subset \mathfrak{m}_N$ and $I_N \subset \mathfrak{m}_N$, so f is an isomorphism by the remark following lemma 0.6. □

Proposition 0.9. *Let $\tau = \tau_{\zeta_1,\zeta_2}$. Then*

$$I(\tau) = (A + D, BT, CT, CQ + BR, A^2 + BC).$$

Proof. Write $\mu = \zeta_1 + \zeta_2 - 2$. The condition that Σ has characteristic polynomial $(X - \zeta_1)(X - \zeta_2)$ is equivalent to the equations

$$A + D = \mu$$
$$A(A - \mu) + BC = \mu.$$

As $X^q \equiv X \mod (X-\zeta_1)(X-\zeta_2)$, we have by the Cayley–Hamilton theorem that $\Sigma^q = \Sigma$ on $\mathcal{A}(\tau)$. The equation $\Phi\Sigma = \Sigma^q\Phi$ therefore becomes $\Phi\Sigma = \Sigma\Phi$, and comparing matrix entries we get three equations (the fourth being redundant):

$$\begin{aligned} BR - CQ &= 0 \\ Q(2A-\mu) &= B(2P-T) \\ R(2A-\mu) &= C(2P-T). \end{aligned}$$

Let

$$J = (A+D-\mu, A(A-\mu)+BC-\mu, BR-CQ, Q(2A-\mu)-B(2P-T), R(2A-\mu)-C(2P-T)).$$

Let I be the image of J in $\overline{\mathcal{S}}$, so that

$$I = \left(A+D, BT, CT, CQ+BR, A^2+BC\right).$$

We have shown that there is a surjection $\mathcal{S}/J \twoheadrightarrow \mathcal{A}(\tau)$, and therefore there is a surjection $f : \overline{\mathcal{S}}/I \to \overline{\mathcal{A}}(\tau)$. We have to show that f is an isomorphism. Write $\tilde{R} = \overline{\mathcal{S}}/I$.

Then (see the proof of corollary 0.10 below) $\overline{\mathcal{S}}/I$ is Cohen–Macaulay, with minimal primes I_1 and I_N, and it is easy to see that $e_{\tilde{R}}(\tilde{R}, I_N) = 1$ while $e_{\tilde{R}}(\tilde{R}, I_1) = 2$.

Let $\rho_1 : G_F \to GL_2(\mathcal{O})$ be diagonal such that the eigenvalues of $\rho_1(\sigma)$ are ζ_1 and ζ_2, and the eigenvalues of $\rho_1(\phi)$ are distinct modulo ϖ. Let $\rho_N : G_F \to GL_2(\mathcal{O})$ send $\sigma \mapsto \begin{pmatrix} \zeta_1 & 1 \\ 0 & \zeta_2 \end{pmatrix}$ and $\phi \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $\mathfrak{m}_1$ and $\mathfrak{m}_N$ be the corresponding maximal ideals of $\overline{\mathcal{A}}(\tau)$. Then $I_1 \subset \mathfrak{m}_1$, $I_N \not\subset \mathfrak{m}_1$, $I_1 \not\subset \mathfrak{m}_N$ and $I_N \subset \mathfrak{m}_N$. By [Sho16b] Proposition 5.3, which remains valid when $p = 2$, $R^{\square}_{\overline{\rho}_1}(\tau)$ is formally smooth over

$$\frac{\mathcal{O}[[X-1]]}{(X-\zeta_1)(X-\zeta_2)}.$$

Therefore $R^{\square}_{\overline{\rho}_1}(\tau) \otimes k$ has a unique minimal prime $\mathfrak{q}$ and its multiplicity is 2. By lemmas 0.3 and 0.6, f is an isomorphism. □

Corollary 0.10. *(of propositions 0.5, 0.8 and 0.9) For $\tau = \tau_\xi$, τ_ζ, or τ_{ζ_1,ζ_2}, $\mathcal{A}(\tau)$ is Cohen–Macaulay.*

Proof. Since ϖ is a regular element of $\mathcal{A}(\tau)$, it suffices to prove that $\overline{\mathcal{A}}(\tau)$ is Cohen–Macaulay. This can easily be checked in magma; we sketch an alternative proof by hand. If $\tau = \tau_\xi$, then by proposition 0.5, $I(\tau) = I_N$. But $\overline{\mathcal{S}}/I_N$ is a complete intersection ring of dimension 4, and therefore is Cohen–Macaulay. If $\tau = \tau_\zeta$, then by proposition 0.8, $I(\tau_\xi) = I_1 \cap I_N$. Now, $\overline{\mathcal{S}}/I_1$ and $\overline{\mathcal{S}}/I_N$ are Cohen–Macaulay of dimension 4 (the latter by the previous case), while $\overline{\mathcal{S}}/(I_1+I_N)$ is regular, and so Cohen–Macaulay, of dimension 3. By exercise 18.13 of [Eis95], $\overline{\mathcal{S}}/(I_1 \cap I_N)$ is also Cohen–Macaulay. Finally, if $\tau = \tau_{\zeta_1,\zeta_2}$ then by proposition 0.9, $I(\tau) = (A+D, A^2+BC, BR+CQ, BT, CT)$. Let $I = I(\tau)$. Since $I + (AT) = I_1 \cap I_N$ and $AT \cdot I_1 = 0$, there is an exact sequence of $\overline{\mathcal{S}}/I$-modules

$$\overline{\mathcal{S}}/I_1 \xrightarrow{AT} \overline{\mathcal{S}}/I \longrightarrow \overline{\mathcal{S}}/(I_1 \cap I_N) \to 0.$$

The first map must be injective, since I_1 is prime and $e_{\overline{\mathcal{S}}/I}(\overline{\mathcal{S}}/I, I_1) = 2 > 1 = e_{\overline{\mathcal{S}}/I}(\overline{\mathcal{S}}/(I_1 \cap I_N), I_1)$. Since we have shown that $\overline{\mathcal{S}}/I_1$ and $\overline{\mathcal{S}}/(I_1 \cap I_N)$ are maximal Cohen–Macaulay modules over $\overline{\mathcal{S}}/I$, so is $\overline{\mathcal{S}}/I$ (by [Yos90] Proposition 1.3). □

Since $R^{\square}_{\overline{\rho}}(\tau)$ is a completion of $\mathcal{A}(\tau)$ by lemma 0.3, and a completion of a Cohen–Macaulay ring is Cohen–Macaulay (by [Sta17, Tag 07NX]), we obtain Theorem 0.1.

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